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BOUNDARY REPRESENTATION FOR LOBACHEVSKY SPACES

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For canonical representations on a Lobachevsky space, a description of distributions concentrated at the boundary is given.
Key words: Lobachevsky spaces; canonical representations; distributions; boundary representations

In this paper we extend to Lobachevsky spaces G/K, $G=\mathrm{SO}_0(n-1,1)$, $K=\mathrm{SO}(n-1)$ our results [1], [2], [3] on distributions concentrated at the boundary of the Lobachevsky plane. We use the Klein model of the Lobachevsky space G/K. It is the unit ball $B\colon \langle u,u\rangle<1$ in $\mathbb{R}^{n-1}$ with the fractional linear action (from the right):

$$u\mapsto u\cdot g=\frac{u\alpha+\gamma}{u\beta+\delta},\qquad g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

Here $\langle u,v\rangle$ is the standard inner product in $\mathbb{R}^{n-1}$:

$$\langle u,v\rangle=u_1v_1+\ldots+u_{n-1}v_{n-1}.$$

The boundary $S\colon \langle u,u\rangle=1$ is the absolute of the Lobachevsky space. Let $\overline{B}=B\cup S$.

Canonical representations R_λ, $\lambda\in\mathbb{C}$, of the group G act on the space $\mathcal{D}(\overline{B})$ by

$$\left(R_\lambda(g)f\right)(u)=f(u\cdot g)(u\beta+\delta)^{-\lambda-n}.$$

They can be extended to the space $\mathcal{D}'(\overline{B})$ of distributions on $\overline{B}$ and, in particular, to the space $\Sigma(\overline{B})$ of distributions concentrated at S. The restriction L_λ of R_λ to $\Sigma(\overline{B})$ is called a *boundary representation.*

Introduce on B polar coordinates $p,s: u=\sqrt{1-p}\cdot s$, $s\in S$, where $p=1-\langle u,u\rangle$. In these coordinates the Laplace–Beltrami operator Δ on B is

$$\Delta=4(1-p)p^2\frac{\partial^2}{\partial p^2}+2p(4-n-3p)\frac{\partial}{\partial p}+\frac{p}{1-p}\,\Delta_S,$$

with the Laplace–Beltrami operator Δ_S on S. Let $\Delta_{\mathfrak{g}}$ be the Casimir element of the Lie algebra $\mathfrak{g}$ of the group G. To this element, the representation R_λ assigns a differential operator (the Casimir operator)

$$\Delta_\lambda=\Delta+4(\lambda+n)p(1-p)\frac{\partial}{\partial p}+(\lambda+n)\big[\lambda+2-(\lambda+n+1)p\big].$$

Its K-radial part (acting on K-invariant functions) is the operator

$$\begin{aligned}\mathrm{Rad}\,\Delta_\lambda &= 4(1-p)p^2\frac{\partial^2}{\partial p^2}+2p\left[2\lambda+n+4-(2\lambda+2n+3)p\right]\frac{\partial}{\partial p}\\ &+ (\lambda+n)\big[\lambda+2-(\lambda+n+1)p\big].\end{aligned}$$

The Berezin form $(f,h)_\lambda$ on $\mathcal{D}(\overline{B})$ is a bilinear form defined by:

$$(f,h)_\lambda = c(\lambda)\int_B \{1-\langle u,v\rangle\}^\lambda f(u)\,h(v)\,du\,dv,$$

где du is the Euclidean measure on B,

$$c(\lambda)=\pi^{-(n-1)/2}\,\Gamma\left(\frac{-\lambda+1}{2}\right)\Big/\Gamma\left(\frac{2-n-\lambda}{2}\right).$$

The Casimir operator and the Berezin form are invariant with respect to R_λ.

Representations T_σ, $\sigma\in\mathbb{C}$, of the group G associated with a cone act on the space $\mathcal{D}(S)$ by

$$\big(T_\sigma(g)\varphi\big)(s)=\varphi(s\cdot g)(s\beta+\delta)^\sigma.$$

They are irreducible for all σ except $\sigma\in\mathbb{N}$ and $\sigma\in 2-n-\mathbb{N}$. Here and further $\mathbb{N}=\{0,1,2,\ldots\}$.

For $\lambda\notin-(n-4)/2+\mathbb{N}$, the boundary representation L_λ decomposes into the direct sum of representations $T_{2-n-\lambda+2k}$, $k\in\mathbb{N}$, as follows. First, we introduce differential operators $W_{\sigma,k}$ and $W^*_{\sigma,k}$ on $\mathcal{D}(S)$, which are polynomials in Δ_S:

$$W_{\sigma,k}=w_k(\sigma,\Delta_S),\quad W^*_{\sigma,k}=w^*_k(\sigma,\Delta_S),$$

where w_k, w_k^* are defined by means of a reproducing function (with $\mu_l=l(3-n-l)$ and the Gauss hypergeometric function F):

$$\begin{aligned}(1-p)^{l/2}\,F\left(\frac{\sigma+n-2+l}{2},\frac{\sigma+n-1+l}{2};\sigma+\frac{n}{2};p\right) &= \sum_{k=0}^{\infty}w_k(\sigma,\mu_l)p^k,\\ (1-p)^{-l/2}\,F\left(\frac{\sigma+2-l}{2},\frac{\sigma+1-l}{2};\sigma+\frac{n}{2};p\right) &= \sum_{k=0}^{\infty}w^*_k(\sigma,\mu_l)p^k.\end{aligned}$$

Then we define operators $\xi_{\lambda,k}:\mathcal{D}(S)\to\Sigma_k(\overline{B})$:

$$\xi_{\lambda,k}(\varphi)=\sum_{b=0}^{k}(-1)^b\,\frac{k!}{(k-b)!}\,W_{\lambda-2k,b}(\varphi)\cdot\delta^{(k-b)}(p),$$

they intertwine $T_{2-n-\lambda+2k}$ with L_λ. The space $\Sigma(\overline{B})$ decomposes into the direct orthogonal (with respect to the Berezin form) sum of their images $V_{\lambda,k}$, $k\in\mathbb{N}$. These subspaces are eigenspaces of Δ_λ with eigenvalues $(\lambda-2k)(\lambda+n-2-2k)$. The "old basis" $\varphi(s)\delta^{(k)}(p)$ is expressed in terms of the "new basis" $\xi_{\lambda,m}(\varphi)$:

$$\varphi(s)\delta^{(k)}(p)=\sum_{r=0}^{k}(-1)^{k-r}\,\frac{k!}{r!}\,\xi_{\lambda,r}\,\left(W^*_{2-n-\lambda+2r,k-r}(\varphi)\right).$$

In the subspace $\Sigma(\overline{B})^K$ of $\Sigma(\overline{B})$ consisting of K-invariant distributions we have two natural bases: the first one consists of derivatives of the delta function $\delta(p)$:

$$\delta^{(k)}(p), \quad k = 0, 1, \ldots, \tag{1}$$

the second one consists of distributions

$$\zeta_{\lambda,k} = \xi_{\lambda,k}(1), \quad k = 0, 1, \ldots . \tag{2}$$

Basis (2) consists of eigenfunctions of Δ_λ and is orthogonal with respect to the Berezin form (we use the notation $a^{[m]} = a(a+1)\ldots(a+m-1)$):

$$\left(\zeta_{\lambda,k}, \zeta_{\lambda,k}\right)_\lambda = \beta(\lambda, k), \quad \left(\zeta_{\lambda,k}, \zeta_{\lambda,r}\right)_\lambda = 0, \quad k \neq r,$$

where

$$\beta(\lambda, k) = b(\lambda) \cdot 2^{-4k}\, k!\, (-1)^k \, \frac{(-\lambda)^{[2k]}(3-n-\lambda)^{[2k]}}{((4-n)/2-\lambda)^{[2k]}\, ((2-n)/2-\lambda+k)^{[k]}}\,,$$

$$b(\lambda) = 2^{\lambda+n-3}\, \pi^{(n-2)/2}\, \frac{\Gamma(\lambda+(n-2)/2)\Gamma((-\lambda+1)/2)}{\Gamma\left((n-1)/2\right)\Gamma((2-n-\lambda)/2)\Gamma(\lambda+n-2)}\,.$$

Elements of bases (1) and (2) are expressed in terms of each other by means of triangular matrices with the unit diagonal, namely,

$$\zeta_{\lambda,k} = \sum_{b=0}^{k} (-1)^b \binom{k}{b} 2^{-2b}\, \frac{(\lambda+n-2-2k)^{[2b]}}{(\lambda+n/2-2k)^{[b]}}\, \delta^{(k-b)}(p)\,, \tag{3}$$

$$\delta^{(k)}(p) = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} 2^{2s-2k}\, \frac{(3-n-\lambda+2s)^{[2k-2s]}}{((4-n)/2-\lambda+2s)^{[k-s]}}\, \zeta_{\lambda,s}\,. \tag{4}$$

Notice that formula (3) can be written as follows:

$$\zeta_{\lambda,k} = F\left(\frac{\lambda+n-2}{2}-k,\; \frac{\lambda+n-1}{2}-k;\; \lambda+\frac{n}{2}-2k;\; p\right)\delta^{(k)}(p).$$

Formula (4) gives a generating function for $\delta^{(k)}(p)$:

$$\exp\left(u\frac{d}{dp}\right)\delta(p) = \sum_{m=0}^{\infty} \frac{u^m}{m!} \times$$
$$\times \quad F\left(\frac{4-n-\lambda}{2}+m,\; \frac{3-n-\lambda}{2}+m;\; \frac{4-n}{2}-\lambda+2m;\; -u\right)\cdot \zeta_{\lambda,m}.$$

Pairwise inner products of basis (1) are given by:

$$\left(\delta^{(m)}(p), \delta^{(r)}(p)\right)_\lambda = b(\lambda)\cdot(-1)^{m+r}\, 2^{-2r-2m}\, \frac{(3-n-\lambda)^{[2m]}(3-n-\lambda)^{[2r]}}{((4-n)/2-\lambda)^{[m+r]}}.$$

REFERENCES

1. *Grosheva L.I.* Representations on distributions on the Lobachevsky plane concentrated at the boundary // Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Review. Series: Natural and Technical Sciences, 1998. V. 3. Iss. 1. P. 46–50.

2. *Grosheva L.I.* Boundary representations on the Lobachevsky plane // Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Review. Series: Natural and Technical Sciences, 2005. V. 10. Iss. 4. P. 357–365.

3. *Molchanov V.F., Grosheva L.I.* Canonical and boundary representations on the Lobachevsky plane // Acta Applicandae Mathematicae. 2002. V. 73. № 1&2. P. 59–77.

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ГРАНИЧНЫЕ ПРЕДСТАВЛЕНИЯ НА ПРОСТРАНСТВЕ ЛОБАЧЕВСКОГО

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Дано описание обобщенных функций, сосредоточенных на границе пространства Лобачевского.
Ключевые слова: пространство Лобачевского; канонические представления; обобщенные функции; граничные представления

СПИСОК ЛИТЕРАТУРЫ

1. *Грошева Л.И.* Представления в обобщенных функциях на плоскости Лобачевского, сосредоточенных на границе // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 1998. Т. 3. Вып. 1. С. 46–50.
2. *Грошева Л.И.* Представления в обобщенных функциях на плоскости Лобачевского, сосредоточенных на границе // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 2005. Т. 10. Вып. 4. С. 357–365.
3. *Molchanov V.F., Grosheva L.I.* Canonical and boundary representations on the Lobachevsky plane // Acta Applicandae Mathematicae. 2002. V. 73. № 1&2. P. 59–77.

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